

On signless Laplacian coefficients of bicyclic graphs *

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Abstract

Let G be a graph of order n and $Q_G(x) = \det(xI - Q(G)) = \sum_{i=1}^n (-1)^i \varphi_i x^{n-i}$ be the characteristic polynomial of the signless Laplacian matrix of a graph G . We give some transformations of G which decrease all signless Laplacian coefficients in the set $\mathcal{B}(n)$ of all n -vertex bicyclic graphs. $\mathcal{B}^1(n)$ denotes all n -vertex bicyclic graphs with at least one odd cycle. We show that B_n^1 (obtained from C_4 by adding one edge between two non-adjacent vertices and adding $n - 4$ pendent vertices at the vertex of degree 3) minimizes all the signless Laplacian coefficients in the set $\mathcal{B}^1(n)$. Moreover, we prove that B_n^2 (obtained from $K_{2,3}$ by adding $n - 5$ pendent vertices at one vertex of degree 3) has minimum signless Laplacian coefficients in the set $\mathcal{B}^2(n)$ of all n -vertex bicyclic graphs with two even cycles.

Key words: Signless Laplacian coefficients; TU-subgraph; Bicyclic graph

AMS Classifications: 05C50, 05C07.

1 Introduction

Let G be a simple undirect bicyclic graph. $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For every bicyclic graph G , $|E(G)| = |V(G)| + 1$. Let $d(v_i)$ denote the degree of vertex v_i , and let $D(G) = \text{diag}(d(v_1), d(v_2), \dots, d(v_n))$ be

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the diagonal matrix of G . Furthermore, let $A(G)$ be the adjacent matrix of G . The Laplacian matrix of G is $L(G) = D(G) - A(G)$, and the Laplacian characteristic polynomial is denoted by $L_G(x) = \det(xI - L(G)) = \sum_{k=1}^n (-1)^k c_k x^{n-k}$. The Laplacian coefficients $c_k(G)$ of a graph G can be expressed in terms of subtree structures of G by the following result of Kelmans and Chelnokov [16]. Let F be a spanning forest of G with k components T_1, T_2, \dots, T_s , T_i has $|V(T_i)|$ vertices, let

$$\gamma(F) = \prod_{i=1}^k |V(T_i)|.$$

Theorem 1.1 ([16]) *Let \mathcal{F}_k be the set of all spanning forests of G with exactly k components. Then the Laplacian coefficient $c_{n-k}(G)$ is expressed by $c_{n-k}(G) = \sum_{F \in \mathcal{F}_k} \gamma(F)$.*

Recently, the study on the Laplacian coefficients have attracted much attention. Mohar [17] first investigate the Laplacian coefficients of acyclic graphs under the partial order \preceq . Zhang et al. [21] investigated ordering trees with diameters 3 and 4 by the Laplacian coefficients. Ilić [13] determined the n -vertex tree of fixed diameter which minimizes the Laplacian coefficients. Ilić [14] determined the n -vertex tree with given matching number having the minimum Laplacian coefficients. He and Li [11] studied the ordering of all n -vertex trees with a perfect matching by Laplacian coefficients. Ilić and Ilić [12] studied the n -vertex trees with fixed pendent vertex number and 2-degree vertex number which have minimum Laplacian coefficients. Stevanović and Ilić [19] investigated the Laplacian coefficients of unicyclic graphs. Tan [20] characterized the determined the n -vertex unicyclic graph with given matching number which minimizes all Laplacian coefficients. He and Shan [10] studied the Laplacian coefficients of bicyclic graphs.

The signless Laplacian matrix of G , $Q(G) = D(G) + A(G)$, which is related to $L(G)$, has also been studied recently (see [1-5, [18]]). The signless Laplacian characteristic polynomial is denoted by $Q_G(x) = \det(xI - Q(G)) = \sum_{i=1}^n (-1)^i \varphi_i x^{n-i}$. Using the notation from [2], [18], a TU-subgraph of G is the spanning subgraph of G whose components are trees or odd unicyclic graphs. Assume that a TU-subgraph H of G contains c odd unicyclic graphs and s trees T_1, \dots, T_s . The weight of H can be expressed by $W(H) = 4^c \prod_{i=1}^s n_i$, in which n_i is the number of T_i . If H contains no tree, let $W(H) = 4^c$. If H is empty, in other words, H does not exist, let $W(H) = 0$. The signless Laplacian coefficients $\varphi_i(G)$ can be expressed in terms of the weight of TU-subgraphs of G .

Theorem 1.2 ([2],[18]) *Let G be a connected graph. For φ_i as above, we have $\varphi_0 = 1$ and*

$$\varphi_i = \sum_{H_i} W(H_i), i = 1, \dots, n;$$

where the summation runs over all TU-subgraphs H_i of G with i edges.

From Theorem 1.2, it is obvious that for a n -vertex connected bicyclic graph G , $\varphi_1(G) = 2|E(G)| = 2(n+1)$.

When G is non-bipartite graph, then G has at least an odd cycle C_1 . Every TU-subgraph of G with n edges is obtained by deleting the edges of $E(C_2) \setminus (E(C_1) \cap E(C_2))$. Therefore,

$$\varphi_n(G) = \begin{cases} |E(C_2) \setminus (E(C_1) \cap E(C_2))|, & \text{if } g(C_2) \text{ is even} \\ \sum_{i=1}^2 |E(C_i) \setminus (E(C_1) \cap E(C_2))|, & \text{if } g(C_2) \text{ is odd.} \end{cases}$$

When G is bipartite graph, G has no odd cycle, then $\varphi_n(G) = 0$, and $\varphi_{n-1}(G)$ counts the number of all spanning trees of G . Every TU-subgraph of G with $n-1$ edges is obtained by deleting one edge e_1 of C_1 and one edge e_2 of C_2 ($e_1 \neq e_2$), respectively. Thus,

$$\varphi_{n-1}(G) = \begin{cases} |E(C_1)||E(C_2)| - |E(C_1 \cap C_2)|(|E(C_1 \cap C_2)| - 1), & \text{if } |E(C_1 \cap C_2)| \geq 1 \\ |E(C_1)||E(C_2)|, & \text{if } |E(C_1 \cap C_2)| = 0. \end{cases}$$

Moreover, $L(G)$ and $Q(G)$ have the same characteristic polynomial, so $c_i(G) = \varphi_i(G)$, $i = 0, 1, 2, \dots, n$, and the expression of φ_i in Theorem 1.2 is equivalence to the expression of c_i in Theorem 1.1.

The eigenvalues of $L(G)$ and $Q(G)$ are denoted by $\mu_1(G) \geq \dots \geq \mu_n(G) = 0$ and $\nu_1(G) \geq \dots \geq \nu_n(G)$, respectively. The incidence energy of G , $IE(G)$ for short, is defined as $IE(G) = \sum_{i=1}^n \sqrt{\nu_i(G)}$ (see [7],[8],[15]).

Mirzakhah and Kiani [18] presented a connection between the incidence energy and the signless Laplacian coefficients.

Theorem 1.3 ([18]) *Let G and G' be two graphs of order n . If $\varphi_i(G) \leq \varphi_i(G')$ for $1 \leq i \leq n$, then $IE(G) \leq IE(G')$ and $IE(G) < IE(G')$ if $\varphi_i(G) < \varphi_i(G')$ for some i holds.*

Mirzakhah and Kiani in [18] gave some results about the signless Laplacian coefficients of a graph G and ordered unicyclic graphs with fixed girth based on the

signless Laplacian coefficients. He and Shan in [10] characterize the graph which has minimum Laplacian coefficients among all bicyclic graphs. Motivated by these results, we characterize the graphs which have minimum signless Laplacian coefficients in $\mathcal{B}^1(n)$ and $\mathcal{B}^2(n)$.

This paper is organized as follows: In the next section, we introduce some results from the literature which are useful in this paper. In Section 3, several transformations which simultaneously decrease all the signless Laplacian coefficients are given. In Section 4, we order the graphs in several sets, and in each set all graphs have the same bases. In Section 5, by using the results of Section 3 and 4, we prove that B_n^1 has minimum signless Laplacian coefficients in $\mathcal{B}^1(n)$, as well as incidence energy. Meanwhile B_n^2 minimizes all the signless Laplacian coefficients and incidence energy in the set $\mathcal{B}^2(n)$.

2 Preliminaries

Let G be a graph which is not a star, let v be a vertex with degree $p+1$ in G , such that it is adjacent with $\{u, v_1, v_2, \dots, v_p\}$, where $\{v_1, v_2, \dots, v_p\}$ are pendent vertices. The graph $G' = \sigma(G, v)$ is obtained from deleting edges vv_1, vv_2, \dots, vv_p and adding edges uv_1, uv_2, \dots, uv_p .

Theorem 2.1 ([18]) *Let G be a connected graph and $G' = \sigma(G, v)$, then $\varphi_i(G) \geq \varphi_i(G')$, for every $0 \leq i \leq n$, with equality if and only if either $i \in \{0, 1, n\}$ when G is non-bipartite, or $i \in \{0, 1, n-1, n\}$ otherwise.*

Let $G = G_1|u : G_2|v$ be the graph obtained from two disjoint graphs G_1 and G_2 by joining a vertex u of G_1 and a vertex v of G_2 by an edge. For any graph G and $v \in V(G)$, let $L_{G|v}(x)$ be the principal submatrix of $L_G(x)$ obtained by deleting the row and column corresponding to the vertex v .

Theorem 2.2 ([6]) *If $G = G_1|u : G_2|v$, then $L_G(x) = L_{G_1}(x)L_{G_2}(x) - L_{G_1}(x)L_{G_2|v}(x) - L_{G_2}(x)L_{G_1|u}(x)$.*

Theorem 2.3 ([9]) *If G be a connected graph with n vertices which consists of a subgraph H ($V(H) \geq 2$) and $n - |V(H)|$ pendent vertices attached to a vertex v in H , then $L_G(x) = (x-1)^{(n-|V(H)|)}L_H(x) - (n-|V(H)|)x(x-1)^{(n-|V(H)|-1)}L_{H|v}(x)$.*

Throughout this paper, we use the following notations. Let $\mathcal{B}(n)$ denote all bicyclic graphs with n vertices. For every graph $G \in \mathcal{B}(n)$, the lengths of the two minimal cycles C_1, C_2 of G is denoted by $g(C_1), g(C_2)$, written by g_1, g_2 for short. It is obvious that $g_1 = |V(C_1)|, g_2 = |V(C_2)|$. Let

$$\mathcal{B}^1(n) = \{G | G \in \mathcal{B}(n), \text{ at least one of } g_1, g_2 \text{ is odd} \},$$

$$\mathcal{B}^2(n) = \{G | G \in \mathcal{B}(n), \text{ both } g_1 \text{ and } g_2 \text{ are even} \}.$$

Let B_n^1 denote the graph obtained from C_4 by adding one edge between two non-adjacent vertices and adding $n - 4$ pendent vertices at the vertex of degree 3, and B_n^2 denote the graph obtained from $K_{2,3}$ by adding $n - 5$ pendent vertices at one vertex of degree 3, where C_4 is a cycle with 4 vertices and $K_{2,3}$ is a complete bipartite graph with 2 and 3 vertices in the two sets, respectively.

Using the notations in [9] and [10], we divide $\mathcal{B}(n)$ into three types. Let \overline{G} denote the base of G , which is the minimal bicyclic subgraph of G . It is easy to see that \overline{G} can be obtained from G by deleting pendent vertices consecutively. Let $B(p, q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by identifying vertex u of C_p and vertex v of C_q . Let $B(p, l, q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by joining vertex u of C_p and vertex v of C_q by a path $uu_1u_2 \cdots u_{l-1}v$ of length l ($l \geq 1$). Let $B(P_k, P_l, P_m)$ ($m \leq l \leq k$) be the bicyclic graph obtained from three pairwise internal disjoint paths of lengths k, l, m from vertices x to y . (see fig.1). Define $\mathcal{B}(n) = \mathcal{B}_1(n) \cup \mathcal{B}_2(n) \cup \mathcal{B}_3(n)$, where

$$\mathcal{B}_1(n) = \{G | G \in \mathcal{B}(n), \overline{G} = B(p, q), p \geq 3, q \geq 3\},$$

$$\mathcal{B}_2(n) = \{G | G \in \mathcal{B}(n), \overline{G} = B(p, l, q), p, q \geq 3, l \geq 1\},$$

$$\mathcal{B}_3(n) = \{G | G \in \mathcal{B}(n), \overline{G} = B(P_k, P_l, P_m), 1 \leq m \leq l \leq k\}.$$

3 Transformations

A pendent edge is an edge which is incident to a vertex of degree 1. Let $N_G(v)$ denote the neighbors of v in the graph G . In this paper, we only consider the cycles of minimal lengths in G and denote the cycles C_1, C_2, \dots . Write $G_1 \leq G$, if G_1 is a subgraph of G . If two cycles C_1, C_2 of G has the form $B(P_k, P_l, P_m)$, then assume $|V(C_1) \cap V(C_2)| = \min\{k, l, m\}$.

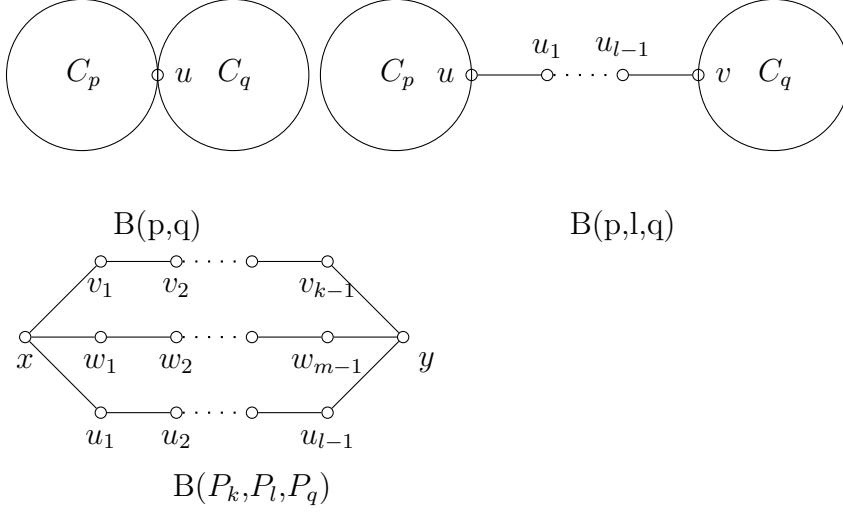


Figure 1: Three types of bases of bicyclic graphs

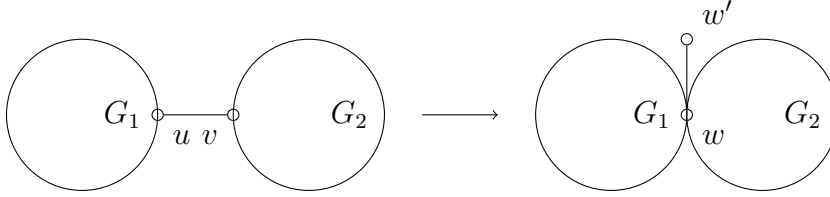


Figure 2: Transformation in Definition 3.1

Definition 3.1 Let G be a simple connected graph with n vertices, and let uv be a nonpendent edge which is not contained in the cycle of G , let G_{uv} obtained from G by identifying vertices u and v and add a new pendent edge ww' to the new vertex w . (see fig.2).

Theorem 3.2 Let G be a n -vertex connected graph, let G and G_{uv} be the two graphs presented in definition 2.1. Then

$$\varphi_i(G) \geq \varphi_i(G_{uv}), i = 0, 1, \dots, n,$$

with equality if and only if either $i \in \{0, 1, n\}$ when G is non-bipartite, or $i \in \{0, 1, n-1, n\}$ otherwise.

Proof. From Theorem 1.2, according to the previous section, we have

$$\varphi_0(G) = \varphi_0(G_{uv}), \varphi_1(G) = \varphi_1(G_{uv}).$$

Since this transformation does not change the length of the cycles, thus, $\varphi_n(G) = \varphi_n(G_{uv})$, and when G is bipartite, $\varphi_{n-1}(G) = \varphi_{n-1}(G_{uv})$.

When G is non-bipartite, for $2 \leq i \leq n-1$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G_{uv} and G with exactly i edges, respectively. For an arbitrary TU-subgraph $H' \in \mathcal{H}'_i$, let R' be the component of H' containing w . Let $N_{R'}(w) \cap N_G(u) = \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$, where $0 \leq r \leq \min\{d_G(u) - 1, |V(R')| - 1\}$, $N_{R'}(w) \cap N_G(v) = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, where $0 \leq s \leq \min\{d_G(v) - 1, |V(R')| - 1\}$. Define H with $V(H) = V(H') - \{w, w'\} + \{u, v\}$, if $ww' \notin E(H')$, $E(H) = E(H') - wu_{i_1} - \dots - wu_{i_r} - wv_{i_1} - \dots - wv_{i_s} + uu_{i_1} + \dots + uu_{i_r} + vv_{i_1} + \dots + vv_{i_s}$. If $ww' \in E(H')$, let $E(H) = E(H') - wu_{i_1} - \dots - wu_{i_r} - wv_{i_1} - \dots - wv_{i_s} + uu_{i_1} + \dots + uu_{i_r} + vv_{i_1} + \dots + vv_{i_s} + uv - ww'$. Let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$.

Now we distinguish \mathcal{H}'_i into the following three cases. Denote G_1 the connected component containing u after deleting uv from G , and let G_2 be the connected component containing v after deleting uv from G .

Case 1: $ww' \in H'$, then H and H' have all the components of equal size, thus $W(H) = W(H')$.

Case 2: $ww' \notin H'$, w is in an odd unicyclic component U' of H' , By the symmetry of G_1 and G_2 , without loss of generality, assume the odd cycle of U' is a subgraph of G_1 . Assume U' contains a vertices in $G_2 \setminus \{w\}$ ($a \geq 0$), then $W(H') = 4 \cdot 1 \cdot N$, for some constant value N , $W(H) = 4 \cdot (a + 1) \cdot N$. Thus $W(H) \geq W(H')$.

Case 3: $ww' \notin H'$, w is in a tree T' of H' . Assume T' contains b vertices in $G_1 \setminus \{w\}$ and c vertices in $G_2 \setminus \{w\}$, then $W(H') = (b + c + 1) \cdot 1 \cdot N$, for some constant value N , $W(H) = (b + 1) \cdot (c + 1) \cdot N$. Thus $W(H) \geq W(H')$.

Therefore, by above discussions, $\varphi_i(G) > \varphi_i(G_{uv})$, $i = 2, \dots, n-1$ holds.

When G is bipartite, it is easy to prove $\varphi_i(G) > \varphi_i(G_{uv})$, $i = 2, \dots, n-2$ by using above discussions of Case 1 and Case 3. ■

Remark. When the subgraph induced by $V(G_2)$ is a star, it is easy to see that the result of Theorem 2.1 is a special case of Theorem 3.2.

Using the transformation of Definition 3.1 consecutively, every graph in $\mathcal{B}_2(n)$ can be transformed into a graph which belongs to $\mathcal{B}_1(n)$, and keep all the signless Laplacian coefficients not increased. Thus the graph which has minimum signless Laplacian coefficients must belong to $\mathcal{B}_1(n)$ or $\mathcal{B}_3(n)$.

Definition 3.3 Let $G = (V, E)$ be a connected graph with at least one cycle C_1 ($V(C_1) \geq 5$). Let $u, v, w \in V(C_1)$ and $u \sim v, v \sim w$. Assume $N_G(u) = \{v, u_1, u_2, \dots\}$,

$N_G(v) = \{u, w, v_1, v_2, \dots\}$, $N_G(w) = \{v, w_1, w_2, \dots\}$, and $N_G(u) \cap N_G(v) = \emptyset$, $N_G(v) \cap N_G(w) = \emptyset$, $N_G(u) \cap N_G(w) = \emptyset$, then the graph

$$G' = G - \{vw, ww_1, ww_2, \dots, vv_1, vv_2, \dots\} + \{uw, uw_1, uw_2, \dots, uv_1, uv_2, \dots\}.$$

Theorem 3.4 *Let $G = (V, E)$ be a connected graph with at least one cycle C_1 ($V(C_1) \geq 5$). Let $u, v, w \in V(C_1)$ and $u \sim v, v \sim w$ as defined in Definition 3.3. If the following statements hold:*

(1). *If $\exists C_2, C_2 \leq G$, such that $|V(C_1) \cap V(C_2)| \leq 2$, then u, v, w satisfy $|\{u, v, w\} \cap V(C_2)| = 0$, or 1.*

(2). *If $\exists C_3, C_3 \leq G$, such that $|V(C_1) \cap V(C_3)| = 3$, then u, v, w satisfy $|\{u, v, w\} \cap V(C_2)| = 0$, or 3.*

(3). *If $\exists C_4, C_4 \leq G$, such that $|V(C_1) \cap V(C_4)| \geq 4$, then u, v, w satisfy $|\{u, v, w\} \cap V(C_2)| = 3$.*

Then by performing the transformation of Definition 3.3 to u, v, w , $\varphi_i(G) \geq \varphi_i(G')$, $i = 0, 1, \dots, n$, with equality if and only if $i \in \{0, 1\}$.

Proof. $\varphi_0(G) = \varphi_0(G')$, and since this transformation does not change the number of edges of G , so $\varphi_1(G) = \varphi_1(G')$. Next suppose $2 \leq i \leq n$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G' and G with exactly i edges, respectively.

First assume C_2 exists, C_3 and C_4 do not exist, and $|\{u, v, w\} \cap V(C_2)| = 1$, without loss of generality, assume $u \in V(C_1) \cap V(C_2)$ and there is no other cycle which contains v or w except C_1 . Then all of u, v, w belong to exactly one cycle C_1 . In the discussion below, if there is no odd cycle which satisfies some case, then we think this case does not exist.

For an arbitrary TU-subgraph $H' \in \mathcal{H}'_i$, let R' be the component of H' containing u . Let $N_{R'}(u) \cap N_G(w) = \{w_{i_1}, w_{i_2}, \dots, w_{i_r}\}$, where $0 \leq r \leq \min\{d_G(w) - 1, |V(R')| - 1\}$, $N_{R'}(u) \cap N_G(v) = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, where $0 \leq s \leq \min\{d_G(v) - 2, |V(R')| - 1\}$. Define H with $V(H) = V(H')$, if $uw \notin E(H')$, $E(H) = E(H') - uw_{i_1} - \dots - uw_{i_r} - uv_{i_1} - \dots - uv_{i_s} + ww_{i_1} + \dots + ww_{i_r} + vv_{i_1} + \dots + vv_{i_s}$. If $uw \in E(H')$, $E(H) = E(H') - uw_{i_1} - \dots - uw_{i_r} - uv_{i_1} - \dots - uv_{i_s} + ww_{i_1} + \dots + ww_{i_r} + vv_{i_1} + \dots + vv_{i_s} - uw + vw$. Let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$.

For convenience, write \mathcal{H}'_i as \mathcal{H}' , and \mathcal{H}_i as \mathcal{H} . If we include u, v, w in a component of H' , then we have components of equal sizes in both TU-subgraphs H' and H , and thus $W(H) = W(H')$ in these cases. Denote $\mathcal{H}^{(0)} = \{H' | uv \in H', vw \in H'\}$. Now we can assume that u, v, w belong to 2 or 3 components.

We distinguish \mathcal{H}' into the following three cases.

Case 1: u is not in an odd unicyclic component of H' . $H = f(H')$. Assume $u \in T'_1$, and there are $a_1 + 1$ vertices in the component containing u in $H - uv$ and $a_2 + 1$ vertices in the component containing w in $H - vw$, $a_3 + 1$ vertices in the component containing v in $H - uv - vw$ ($a_1, a_2, a_3 \geq 0$). Denote N be the product of all the orders of components of H' except the components containing u, v, w .

Subcase 1.1: $uv \in H', uw \notin H'$, then $W(H') = (a_1 + a_2 + a_3 + 2) \cdot 1 \cdot N$, for some constant value N . $W(H) = (a_1 + a_3 + 2) \cdot (a_2 + 1) \cdot N$, so $W(H) - W(H') = [a_2 \cdot (a_1 + a_3 + 1)] \cdot N \geq 0$. Denote $\mathcal{H}'^{(11)} = \{H' | u \in T'_1, uv \in H', uw \notin H'\}$. Then $\sum_{H' \in \mathcal{H}'^{(11)}} [W(H) - W(H')] \geq 0$.

Subcase 1.2: $uv, uw \notin H'$, $W(H') = (a_1 + a_2 + a_3 + 1) \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (a_1 + 1) \cdot (a_2 + 1) \cdot (a_3 + 1) \cdot N$, so $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'^{(12)} = \{H' | u \in T'_1, uv, uw \notin H'\}$. Then $\sum_{H' \in \mathcal{H}'^{(12)}} [W(H) - W(H')] \geq 0$.

Subcase 1.3: $uv \notin H', uw \in H'$, then $W(H') = (a_1 + a_2 + a_3 + 2) \cdot 1 \cdot N$, for some constant value N . $W(H) = (a_2 + a_3 + 2) \cdot (a_1 + 1) \cdot N$, so $W(H) - W(H') = [a_1 \cdot (a_2 + a_3 + 1)] \cdot N \geq 0$. Denote $\mathcal{H}'^{(13)} = \{H' | u \in T'_1, uv \notin H', uw \in H'\}$. Then $\sum_{H' \in \mathcal{H}'^{(13)}} [W(H) - W(H')] \geq 0$.

Case 2: u is in an odd unicyclic component U' of H' , and C'_1 is a subgraph of U' .

Subcase 2.1: $uv, uw \notin H'$, then $W(H') = 4 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) \geq (g(C_1) - 1) \cdot 1 \cdot N$, so $W(H) - W(H') \geq (g(C_1) - 5) \cdot N \geq 0$ by $g(C_1) \geq 5$. Denote $\mathcal{H}'^{(21)} = \{H' | u \in U', uv, uw \notin H'\}$, then $\sum_{H' \in \mathcal{H}'^{(21)}} [W(H) - W(H')] \geq 0$.

Subcase 2.2: $uv \notin H', uw \in H'$ or $uw \notin H', uv \in H'$, then $W(H') = 4 \cdot 1 \cdot N$, for some constant value N . $W(H) \geq g(C_1) \cdot N$, so $W(H) - W(H') \geq (g(C_1) - 4) \cdot N > 0$ by $g(C_1) \geq 5$. Denote $\mathcal{H}'^{(22)} = \{H' | u \in U', uv \notin H', uw \in H' \text{ or } uw \notin H', uv \in H'\}$, then $\sum_{H' \in \mathcal{H}'^{(22)}} [W(H) - W(H')] > 0$.

Case 3: u is in an odd unicyclic component U' of H' , and C'_1 is not a subgraph of U' . Without loss of generality, assume the subgraph C_2 of G is a subgraph of U' .

Subcase 3.1: $uv, uw \notin H'$, then $W(H') = 4 \cdot 1 \cdot 1 \cdot N$, for some constant value N . Assume the order of the tree in H containing v, w is b_1, b_2 ($b_1, b_2 \geq 1$), respectively. $W(H) = 4 \cdot b_1 \cdot b_2 \cdot N$, so $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'^{(31)} = \{H' | u \in U', uv, uw \notin H'\}$, then $\sum_{H' \in \mathcal{H}'^{(31)}} [W(H) - W(H')] \geq 0$.

Subcase 3.2: $uv \notin H', uw \in H'$ or $uw \notin H', uv \in H'$, then $W(H') = 4 \cdot 1 \cdot N$, for some constant value N . Assume the order of the tree T in H containing w is c . Since $v \in T$, thus $c \geq 1$. $W(H) = 4 \cdot c \cdot N$, so $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'^{(32)} = \{H' | u \in U', uv \notin H', uw \in H' \text{ or } uw \notin H', uv \in H'\}$, then $\sum_{H' \in \mathcal{H}'^{(32)}} [W(H) - W(H')] \geq 0$.

Thus by summing over all possible subsets of \mathcal{H}'_i , ($\mathcal{H}'_i = \mathcal{H}'^{(0)} \cup \mathcal{H}'^{(11)} \cup \mathcal{H}'^{(12)} \cup \mathcal{H}'^{(13)} \cup \mathcal{H}'^{(21)} \cup \mathcal{H}'^{(22)} \cup \mathcal{H}'^{(31)} \cup \mathcal{H}'^{(32)}$), from Theorem 1.2 and f is an injection on the whole. Then

$$\varphi_i(G') = \sum_{H' \in \mathcal{H}'_i} W(H') < \sum_{H \in \mathcal{H}_i^*} W(H) \leq \sum_{H \in \mathcal{H}_i} W(H) = \varphi_i(G)$$

holds for $i = 2, 3, \dots, n-1, n$.

For other cases in which u, v, w satisfy, the discussion is similar, thus we omit it.

■

Remark. When C_2 exists, and $|\{u, v, w\} \cap V(C_2)| = 0$, or 1, then after performing transformation in Definition 3.3, $g(C'_1) = g(C_1) - 2, g(C'_2) = g(C_2)$.

When C_3 exists, and $|\{u, v, w\} \cap V(C_3)| = 0$, then after performing transformation in Definition 3.3, $g(C'_1) = g(C_1) - 2, g(C'_3) = g(C_3)$.

When C_3 (resp. C_4) exists, and $|\{u, v, w\} \cap V(C_3)| = 3$ (resp. $|\{u, v, w\} \cap V(C_4)| = 3$), then after performing transformation in Definition 3.3, $g(C'_1) = g(C_1) - 2, g(C'_3) = g(C_3) - 2$ (resp. $g(C'_4) = g(C_4) - 2$).

4 The ordering of graphs in seven special sets

For convenience, we define $B(3, 3)$ as B_1 , and $V(B_1) = \{x, u, v, w, z\}$, where $d(u) = d(v) = d(w) = d(z) = 2, d(x) = 3$. The graph $B_1(a, b, c, d, e)$ is obtained from B_1 by adding a, b, c, d, e pendent vertices at vertices x, u, v, w, z , respectively.

We define $B(3, 4)$ as B_2 , and $V(B_2) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, where $d(u_2) = d(u_3) = d(u_4) = d(u_5) = d(u_6) = 2, d(u_1) = 4$. The graph $B_2(a, b, c, d, e, f)$ is obtained from B_2 by adding a, b, c, d, e, f pendent vertices at vertices $u_1, u_2, u_3, u_4, u_5, u_6$, respectively.

We define $B(P_2, P_2, P_1)$ as B_3 , and $V(B_3) = \{u, v, w, z\}$, where $d(w) = d(z) = 2, d(u) = d(v) = 3$. The graph $B_3(a, b, c, d)$ is obtained from B_3 by adding a, b, c, d pendent vertices at vertices u, v, w, z , respectively.

We define $B(P_3, P_2, P_1)$ as B_4 , and $V(B_4) = \{u, v, w, z, x\}$, where $d(w) = d(z) = d(x) = 2, d(u) = d(v) = 3$. The graph $B_4(a, b, c, d, e)$ is obtained from B_4 by adding a, b, c, d, e pendent vertices at vertices u, v, w, z, x , respectively.

We define $B(P_3, P_2, P_2)$ as B_5 , and $V(B_5) = \{u, v, u_1, v_1, w_1, w_2\}$, where $d(u) = d(v) = 3, d(u_1) = d(v_1) = d(w_1) = d(w_2) = 2$. The graph $B_5(a, b, c, d, e, f)$ is ob-

tained from B_5 by adding a, b, c, d, e, f pendent vertices at vertices u, v, u_1, v_1, w_1, w_2 , respectively. (See fig.3).

Lemma 4.1 *Let $B_1(a, b, c, d, e)$ be the graph defined above, if we move all pendent edges from vertices u, v, w, z to vertex x . If $a + b + c + d \neq 0$, then*

$$\varphi_i(B_1(a, b, c, d, e)) \geq \varphi_i(B_1(a + b + c + d + e, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n - 1, n\}$.

Proof. The equality $\varphi_i(B_1(a, b, c, d, e)) = \varphi_i(B_1(a + b + c + d + e, 0, 0, 0, 0)), i = 0, 1, n - 1, n$ can be proved as the proof of Theorem 3.2.

For $2 \leq i \leq n - 2$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G' and G with exactly i edges, respectively. Let $\mathcal{H}'_i = \mathcal{H}'_i{}^1 \cup \mathcal{H}'_i{}^2 \cup \mathcal{H}'_i{}^3 \cup \mathcal{H}'_i{}^4 \cup \mathcal{H}'_i{}^5$, where $\mathcal{H}'_i{}^j (j = 1, 2, 3, 4, 5)$ denotes vertices u, v, w, z, x belong to exactly j components. \mathcal{H}_i^j can be defined similarly.

For an arbitrary $H' \in \mathcal{H}'_i$, assume x is in a component R' of H' . Denote a_1 (resp. b_1, c_1, d_1, e_1) be the number of isolated vertices in the set $N_{G'}(x) \cap N_G(u)$ (resp. $N_{G'}(x) \cap N_G(v), N_{G'}(x) \cap N_G(w), N_{G'}(x) \cap N_G(z), N_{G'}(x) \cap (N_G(u) \setminus \{v, z\})$). Write $A = a + 1 - a_1, B = b + 1 - b_1, C = c + 1 - c_1, D = d + 1 - d_1, E = e + 1 - e_1$, without loss of generality, assume $N_G(u) \cap N_{R'}(x) = \{u^1, \dots, u^{b-b_1}\}, N_G(v) \cap N_{R'}(x) = \{v^1, \dots, v^{c-c_1}\}, N_G(w) \cap N_{R'}(x) = \{w^1, \dots, w^{d-d_1}\}, N_G(z) \cap N_{R'}(x) = \{z^1, \dots, z^{e-e_1}\}$, and $(N_G(x) \setminus \{v, z\}) \cap N_{R'}(x) = \{x^1, \dots, x^{a-a_1}\}$. Define H with $V(H) = V(H')$,

$$E(H) = E(H') - \{xu^1, \dots, xu^{b-b_1}, xv^1, \dots, xv^{c-c_1}, xw^1, \dots, xw^{d-d_1}, xz^1, \dots, xz^{e-e_1}\} \\ + \{uu^1, \dots, uu^{b-b_1}, vv^1, \dots, vv^{c-c_1}, ww^1, \dots, ww^{d-d_1}, zz^1, \dots, zz^{e-e_1}\}.$$

Then $H \in \mathcal{H}_i$, let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$. It is easy to see that $H' \in \mathcal{H}'_i{}^j \Leftrightarrow H \in \mathcal{H}_i^j, j = 1, 2, 3, 4, 5$.

If we include vertices u, v, w, z, x in a component of H' , we have equal sizes of components in H' and H , respectively. Then $W(H') = W(H)$.

Assume vertices u, v, w, z, x belong to at least two components. We distinguish the proof into two cases.

Case 1: When all components of H' are trees, and denote the set of this kind of H' as $\mathcal{H}'_{(1)}$. Then $\sum_{H' \in \mathcal{H}'_{(1)}} (W(H) - W(H')) \geq 0$, with equality holds if and only if $i \in \{0, 1, n - 1, n\}$ by Lemma 3.4 in [10].

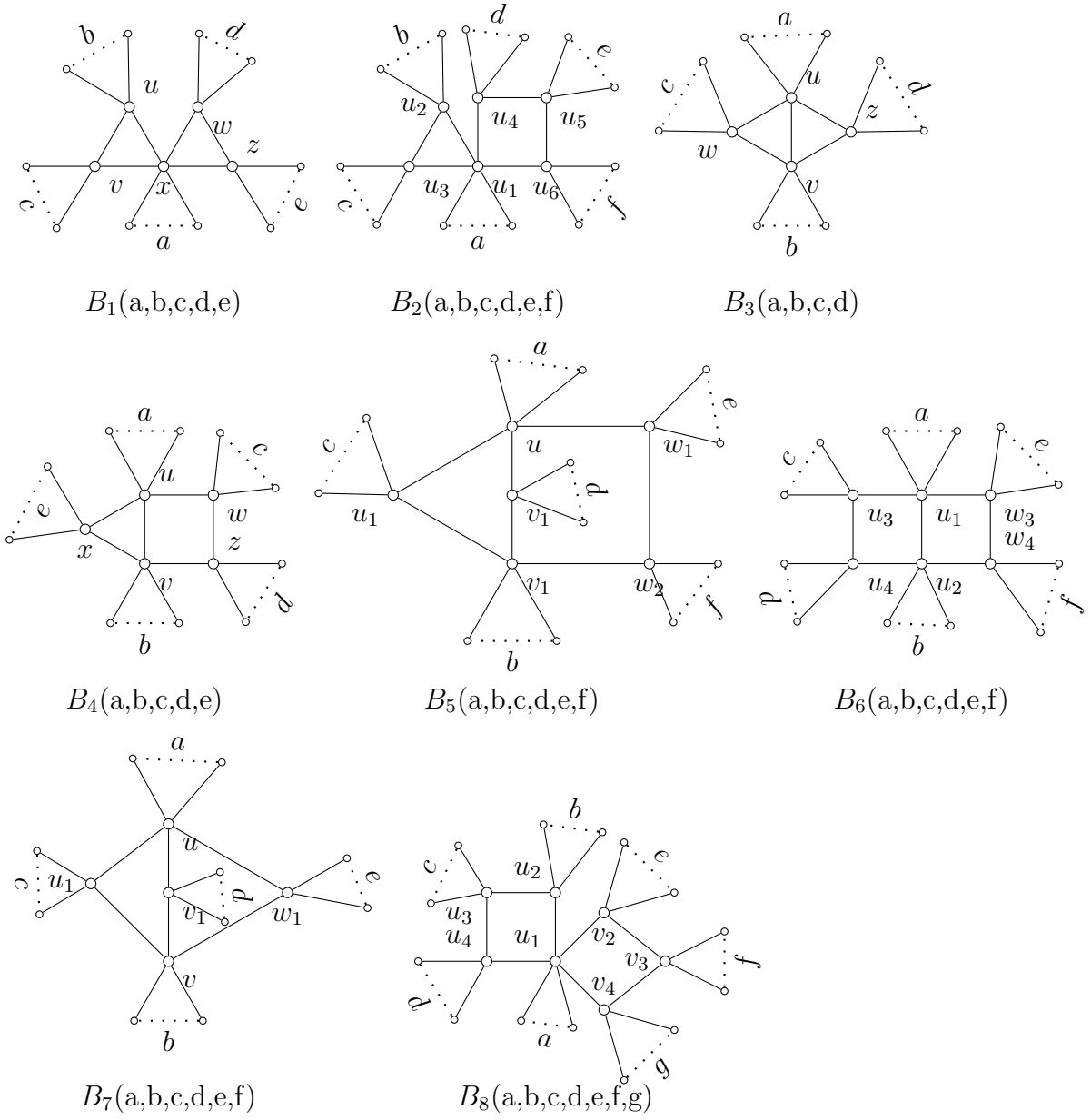


Figure 3: Seven types of bicyclic graphs

Case 2: When vertex x belongs to an odd unicyclic component U' , without loss of generality, assume $C_1 : uvx$ is a subgraph of U' .

Subcase 2.1: Both $\{w, z\}$ belong to U' , then $W(H') = 4, W(H) = 4$. Denote the set of this kind of H' as $\mathcal{H}'_{(21)}$. Then $\sum_{H' \in \mathcal{H}'_{(21)}} (W(H) - W(H')) = 0$.

Subcase 2.2: At most one vertex in $\{w, z\}$ belongs to U' . Denote the set of this kind of H' as $\mathcal{H}'_{(22)}$. When $wx \in H', wz, xz \notin H'$, then $W(H') = 4 \cdot 1, W(H) = 4 \cdot D$, thus $W(H) - W(H') \geq 0$. When $xz \in H', wz, wx \notin H'$, then $W(H') = 4 \cdot 1, W(H) = 4 \cdot C$, thus $W(H) - W(H') \geq 0$. When $wz \in H', wx, xz \notin H'$, then $W(H') = 4 \cdot 2, W(H) = 4 \cdot (C + D)$, thus $W(H) - W(H') \geq 0$. When $wz, wx, xz \notin H'$, then $W(H') = 4 \cdot 1 \cdot 1, W(H) = 4 \cdot C \cdot D$, thus $W(H) - W(H') \geq 0$. Then $\sum_{H' \in \mathcal{H}'_{(22)}} (W(H) - W(H')) \geq 0$.

Thus by summing over all possible subsets of \mathcal{H}'_i , ($\mathcal{H}'_i = \mathcal{H}'_{(1)} \cup \mathcal{H}'_{(21)} \cup \mathcal{H}'_{(22)}$), from Theorem 1.2 and f is an injection on the whole. Then

$$\varphi_i(G') = \sum_{H' \in \mathcal{H}'_i} W(H') < \sum_{H \in \mathcal{H}_i^*} W(H) \leq \sum_{H \in \mathcal{H}_i} W(H) = \varphi_i(G)$$

holds for $i = 2, 3, \dots, n-1$.

■

Similar to the proof of Lemma 4.1, and by Lemma 3.2 and Lemma 3.3 in [10], the following Lemma holds:

Lemma 4.2 (1). *Let $B_2(a, b, c, d, e, f)$ be the graph defined above, if we move all pendent edges from vertices u_2, u_3 to vertex u_1 . If $b + c \neq 0$, then*

$$\varphi_i(B_2(a, b, c, d, e, f)) \geq \varphi_i(B_2(a + b + c, 0, 0, d, e, f)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(2). *Let $B_2(a, 0, 0, d, e, f)$ be the graph defined above, if we move all pendent edges from vertices u_4, u_5, u_6 to vertex u_1 . If $d + e + f \neq 0$, then*

$$\varphi_i(B_2(a, 0, 0, d, e, f)) \geq \varphi_i(B_2(a + d + e + f, 0, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(3). *Let $B_3(a, b, c, d)$ be the graph defined above, if we move all pendent edges from vertices v, w, z to vertex u . If $b + c + d \neq 0$, then*

$$\varphi_i(B_3(a, b, c, d)) \geq \varphi_i(B_3(a + b + c + d, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(4). Let $B_4(a, b, c, d, e)$ be the graph defined above, if we move all pendent edges from vertices x to vertex u . If $e \neq 0$, then

$$\varphi_i(B_4(a, b, c, d, e)) \geq \varphi_i(B_4(a+e, b, c, d, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(5). Let $B_4(a, b, c, d, 0)$ be the graph defined above, if we move all pendent edges from vertices w, z to vertex u . If $b+c \neq 0$, then

$$\varphi_i(B_4(a, b, c, d, 0)) \geq \varphi_i(B_4(a+c+d, b, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(6). Let $B_4(a, b, 0, 0, 0)$ be the graph defined above, if we move all pendent edges from vertices v to vertex u . If $b \neq 0$, then

$$\varphi_i(B_4(a, b, 0, 0, 0)) \geq \varphi_i(B_4(a+b, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(7). Let $B_5(a, b, c, d, e, f)$ be the graph defined above, if we move all pendent edges from vertices $\{u_1, u_2, v_1, v_2, w\}$ to vertex u . If $b+c+d+e+f \neq 0$, then

$$\varphi_i(B_5(a, b, c, d, e, f)) \geq \varphi_i(B_5(a+b+c+d+e+f, 0, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

For convenience, we define $B(P_3, P_3, P_1)$ as B_6 , and $V(B_6) = \{u_1, u_2, u_3, u_4, w_3, w_4\}$, where $d(u_3) = d(u_4) = d(w_3) = d(w_4) = 2, d(u_1) = d(u_2) = 3$. The graph $B_6(a, b, c, d, e, f)$ is obtained from B_6 by adding a, b, c, d, e, f pendent vertices at vertices $u_1, u_2, u_3, u_4, w_3, w_4$, respectively.

We define $B(P_2, P_2, P_2)$ as B_7 , and $V(B_7) = \{u, v, u_1, w_1, v_1\}$, where $d(u_1) = d(v_1) = d(w_1) = 2, d(u) = d(v) = 3$. The graph $B_6(a, b, c, d, e)$ is obtained from B_7 by adding a, b, c, d, e pendent vertices at vertices u, v, u_1, w_1, v_1 , respectively.

We define $B(4, 4)$ as B_8 , and $V(B_8) = \{u_1, u_2, u_3, u_4, v_2, v_3, v_4\}$, where $d(u_2) = d(u_3) = d(u_4) = d(v_2) = d(v_3) = d(v_4) = 2, d(u_1) = 4$. The graph $B_8(a, b, c, d, e, f, g)$ is obtained from B_8 by adding a, b, c, d, e, f, g pendent vertices at $u_1, u_2, u_3, u_4, v_2, v_3, v_4$, respectively. (See fig. 3).

Lemma 4.3 *Let $B_6(a, b, c, d, e, f)$ be the graph defined above, if we move all pendent edges from vertices u_3, u_4 to vertex u_1 . If $c + d \neq 0$, then*

$$\varphi_i(B_6(a, b, c, d, e, f)) \geq \varphi_i(B_6(a + c + d, b, 0, 0, e, f)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n - 1, n\}$.

Proof. The equality $\varphi_i(B_6(a, b, c, d, e, f)) = \varphi_i(B_6(a + c + d, b, 0, 0, e, f)), i = 0, 1, n - 1, n$ can be proved as the proof of Theorem 3.2.

For $2 \leq i \leq n - 2$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G' and G with exactly i edges, respectively. Let $\mathcal{H}'_i = \mathcal{H}'_i{}^1 \cup \mathcal{H}'_i{}^2 \cup \mathcal{H}'_i{}^3 \cup \mathcal{H}'_i{}^4$, where $\mathcal{H}'_i{}^j (j = 1, 2, 3, 4)$ denotes vertices u_1, u_2, u_3, u_4 belong to exactly j components. \mathcal{H}'_i can be defined similarly.

For an arbitrary $H' \in \mathcal{H}'_i$, assume u_1 is in a component R' of H' . Assume $(N_G(u_3) \setminus \{u_2, u_4\}) \cap N_{R'}(u_1) = \{u_3^{i_1}, \dots, u_3^{i_r}\}$, $N_G(u_4) \cap N_{R'}(u_1) = \{u_4^{i_1}, \dots, u_4^{i_s}\}$, where $0 \leq r \leq \min\{d_G(u_3) - 2, |V(R')| - 1\}$, $0 \leq s \leq \min\{d_G(u_4) - 2, |V(R')| - 1\}$. Define H with $V(H) = V(H')$, $E(H) = E(H') - \{u_1u_3^{i_1} - \dots - u_1u_3^{i_r} - u_1u_4^{i_1} - \dots - u_1u_4^{i_s} + u_3u_3^{i_1} + \dots + u_3u_3^{i_r} + u_4u_4^{i_1} + \dots + u_4u_4^{i_s}\}$. Then $H \in \mathcal{H}_i$, let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$. It is easy to see that $H' \in \mathcal{H}'_i{}^j \Leftrightarrow H \in \mathcal{H}_i{}^j, j = 1, 2, 3, 4$.

Let $A + 1$ be the order of the subgraph of $H' - u_1u_2 - u_1u_4$ which contains u_1 and excluding the vertices in $N_G(u_3) \cup N_G(u_4)$ and $B + 1$ be the order of the subgraph of $H' - u_1u_2 - u_2u_3$ which contains u_2 , and denote $C = |N_{R'}(u_1) \cap N_G(u_4)|$, $D = |N_{R'}(u_1) \cap (N_G(u_3) \setminus \{u_2, u_4\})|$, ($A, B, C, D \geq 0$, and A, B, C, D is fixed). Denote N the product of all the orders of components of H' except the components containing u_1, u_2, u_3, u_4 .

If we include vertices u_1, u_3, u_4 in a component of H' , we have equal sizes of components in H' and H , respectively. Then $W(H') = W(H)$.

Assume vertices u_1, u_3, u_4 belong to at least two components. we distinguish the proof into three cases.

Case 1: $H' \in \mathcal{H}'_i{}^2$.

Subcase 1.1: $u_1u_2, u_3u_4 \in H'$, $u_2u_3, u_1u_4 \notin H'$, then

$$W(H) - W(H') = [(A + B + 2)(C + D + 2) - 2(A + B + C + D + 2)] \cdot N.$$

Subcase 1.2: $u_1u_2, u_3u_4 \notin H'$, $u_2u_3, u_1u_4 \in H'$, then

$$W(H) - W(H') = [(A + D + 2)(B + C + 2) - (B + 2)(A + C + D + 2)] \cdot N.$$

Subcase 1.3: $u_1u_2, u_2u_3 \in H', u_3u_4, u_1u_4 \notin H'$, then

$$W(H) - W(H') = [(A + B + C + 3)(D + 1) - (A + B + C + D + 3)] \cdot N.$$

Subcase 1.4: $u_1u_2, u_1u_4 \in H', u_3u_4, u_2u_3 \notin H'$, then

$$W(H) - W(H') = [(A + B + D + 3)(C + 1) - (A + B + C + D + 3)] \cdot N.$$

Subcase 1.5: $u_2u_3, u_3u_4 \in H', u_1u_2, u_1u_4 \notin H'$, then

$$W(H) - W(H') = [(B + C + D + 3)(A + 1) - (A + C + D + 1)(B + 3)] \cdot N.$$

We can find a bijection between every two subsets of the above subcases. Hence

$$\sum_{H \in \mathcal{H}_i^2} W(H) - \sum_{H' \in \mathcal{H}_i'^2} W(H') \geq 0.$$

Case 2: $H' \in \mathcal{H}_i'^3$.

Subcase 2.1: $u_1u_2 \in H', u_2u_3, u_3u_4, u_1u_4 \notin H'$, then

$$W(H) - W(H') = [(A + B + 2)(C + 1)(D + 1) - (A + B + C + D + 2)] \cdot N.$$

Subcase 2.2: $u_2u_3 \in H', u_1u_2, u_3u_4, u_1u_4 \notin H'$, then

$$W(H) - W(H') = [(A + 1)(D + 1)(B + C + 2) - (B + 2)(A + C + D + 1)] \cdot N.$$

Subcase 2.3: $u_3u_4 \in H', u_1u_2, u_2u_3, u_1u_4 \notin H'$, then

$$W(H) - W(H') = [(A + 1)(B + 1)(C + D + 2) - 2(B + 1)(A + C + D + 1)] \cdot N.$$

Subcase 2.4: $u_1u_4 \in H', u_1u_2, u_2u_3, u_3u_4 \notin H'$, then

$$W(H) - W(H') = [(A + D + 2)(B + 1)(C + 1) - (A + C + D + 2)(B + 1)] \cdot N.$$

We can find a bijection between every two subsets of the above subcases. Hence

$$\sum_{H \in \mathcal{H}_i^3} W(H) - \sum_{H' \in \mathcal{H}_i'^3} W(H') \geq 0.$$

Case 3: $H' \in \mathcal{H}_i'^4$, $u_1u_2, u_2u_3, u_3u_4, u_1u_4 \notin H'$, then $W(H) - W(H') = [(A + 1)(B + 1)(C + 1)(D + 1) - (A + C + D + 1)(B + 1)] \cdot N = (B + 1)(ACD + AC + CD + AD) \cdot N$.

Thus

$$\sum_{H \in \mathcal{H}_i^4} W(H) - \sum_{H' \in \mathcal{H}_i'^4} W(H') \geq 0.$$

Combining the above cases and $\sum_{H \in \mathcal{H}_i^1} W(H) = \sum_{H' \in \mathcal{H}_i^1} W(H')$, from Theorem 1.2 and f is an injection on the whole. Then

$$\varphi_i(G') = \sum_{H' \in \mathcal{H}_i'} W(H') < \sum_{H \in \mathcal{H}_i^*} W(H) \leq \sum_{H \in \mathcal{H}_i} W(H) = \varphi_i(G)$$

holds for $i = 2, 3, \dots, n-1$.

■

Similar to the proof of Lemma 4.3, we have the following Lemma.

Lemma 4.4 (1). *Let $B_6(a, b, 0, 0, 0, 0)$ be the graph defined above, if we move all pendent edges from vertices u_2 to vertex u_1 . If $b \neq 0$, then*

$$\varphi_i(B_6(a, b, 0, 0, 0, 0)) \geq \varphi_i(B_6(a + b, 0, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(2). *Let $B_7(a, b, c, d, e)$ be the graph defined above, if we move all pendent edges from vertices u_1 to vertex u . If $c \neq 0$, then*

$$\varphi_i(B_7(a, b, c, d, e)) \geq \varphi_i(B_7(a + c, b, 0, d, e)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(3). *Let $B_7(a, b, 0, 0, 0)$ be the graph defined above, if we move all pendent edges from vertices v to vertex u . If $b \neq 0$, then*

$$\varphi_i(B_7(a, b, 0, 0, 0)) \geq \varphi_i(B_7(a + b, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(4). *Let $B_8(a, b, c, d, e, f, g)$ be the graph defined above, if we move all pendent edges from vertices u_2, u_3, u_4 to vertex u_1 . If $b + c + d \neq 0$, then*

$$\varphi_i(B_8(a, b, c, d, e, f, g)) \geq \varphi_i(B_8(a + b + c + d, 0, 0, 0, e, f, g)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

(5). *Let $B_8(a, 0, 0, 0, e, f, g)$ be the graph defined above, if we move all pendent edges from vertices v_2, v_3, v_4 to vertex u_1 . If $e + f + g \neq 0$, then*

$$\varphi_i(B_8(a, 0, 0, 0, e, f, g)) \geq \varphi_i(B_8(a + e + f + g, 0, 0, 0, 0, 0, 0)), i = 0, 1, \dots, n,$$

with equality holds if and only if $i \in \{0, 1, n-1, n\}$.

For any graph G and $v \in V(G)$, let $Q_{G|v}(x)$ be the principal submatrix of $Q_G(x)$ obtained by deleting the row and column corresponding to the vertex v . Similar to the proof of Theorem 2.2 and Theorem 2.3, we can prove the following two lemmas.

Lemma 4.5 *If $G = G_1|u : G_2|v$, then $Q_G(x) = Q_{G_1}(x)Q_{G_2}(x) - Q_{G_1}(x)Q_{G_2|v}(x) - Q_{G_2}(x)Q_{G_1|u}(x)$.*

Lemma 4.6 *If G be a connected graph with n vertices which consists of a subgraph H ($|V(H)| \geq 2$) and $n - |V(H)|$ pendent vertices attached to a vertex v in H , then $Q_G(x) = (x - 1)^{(n-|V(H)|)}Q_H(x) - (n - |V(H)|)x(x - 1)^{(n-|V(H)|-1)}Q_{H|v}(x)$.*

Let $f(x) = \sum_{i=0}^n (-1)^i a_i x^{n-i}$, $g(x) = \sum_{j=0}^m (-1)^j a_j x^{m-j}$, $a_i > 0, b_j > 0$. Then it is easy to see $f(x)g(x) = f(x) = \sum_{k=0}^{m+n} (-1)^k \sum_{i=0}^k a_i b_{k-i} x^{n+m-k}$ has coefficients alternate with positive and negative.

By using Lemma 4.5 and Lemma 4.6, we can compute the signless Laplacian polynomials of seven special n -vertex bicyclic graphs. For convenience, write $Q_G(x)$ as $Q(G, x)$.

$$\begin{aligned}
Q(B_1(n-5, 0, 0, 0, 0), x) &= (x-1)^{n-4}(x-3)[x^3 - (n+3)x^2 + 3nx - 8], \\
Q(B_2(n-6, 0, 0, 0, 0, 0), x) &= (x-1)^{n-6}(x-2)[x^5 - (n+6)x^4 + 7(n+1)x^3 - 2(7n-1)x^2 + 2(3n+8)x - 8], \\
Q(B_3(n-4, 0, 0, 0), x) &= (x-1)^{n-4}(x-2)[x^3 - (n+4)x^2 + 4nx - 8], \\
Q(B_4(n-5, 0, 0, 0, 0), x) &= (x-1)^{n-6}[x^6 - (n+8)x^5 + 9(n+2)x^4 - (27n+10)x^3 + (31n+10)x^2 - (11n+32)x + 16], \\
Q(B_5(n-6, 0, 0, 0, 0, 0), x) &= (x-1)^{n-7}(x-2)[x^6 - (n+7)x^5 + (9n+8)x^4 - (26n-22)x^3 + (27n-30)x^2 - (8n+8)x + 8], \\
Q(B_6(n-6, 0, 0, 0, 0, 0), x) &= x(x-1)^{n-6}(x-3)[x^4 - (n+5)x^3 + (7n-1)x^2 - (13n-17)x + 5n], \\
Q(B_7(n-5, 0, 0, 0, 0), x) &= x(x-1)^{n-6}(x-2)^2[x^3 - (n+4)x^2 + (5n-2)x - 3n], \\
Q(B_8(n-7, 0, 0, 0, 0, 0, 0), x) &= x(x-1)^{n-8}(x-2)^2(x^2 - 4x + 2)[x^3 - (n+2)x^2 + 2(2n-3)x - 2n].
\end{aligned}$$

Then we have

$$\begin{aligned}
&Q(B_1(n-5, 0, 0, 0, 0), x) - Q(B_3(n-4, 0, 0, 0), x) \\
&= (x-1)^{n-4}(x^2 - nx + 8).
\end{aligned} \tag{1}$$

$$\begin{aligned}
&Q(B_2(n-6, 0, 0, 0, 0, 0), x) - Q(B_4(n-5, 0, 0, 0, 0), x) \\
&= x(x-1)^{n-6}[x^3 - (n+2)x^2 + (3n+2)x - (n+8)].
\end{aligned} \tag{2}$$

$$\begin{aligned}
& Q(B_4(n-5, 0, 0, 0, 0), x) - Q(B_3(n-4, 0, 0, 0), x) \\
&= x(x-1)^{n-6}[(n-3)x^3 - (6n-20)x^2 + (9n-30)x - (3n-8)]. \quad (3)
\end{aligned}$$

$$\begin{aligned}
& Q(B_5(n-6, 0, 0, 0, 0, 0), x) - Q(B_3(n-4, 0, 0, 0), x) \\
&= x(x-2)(x-1)^{n-7}[(2n-7)x^3 - (11n-43)x^2 + (14n-58)x - (4n-16)]. \quad (4)
\end{aligned}$$

Thus $B_3(n-4, 0, 0, 0)$ has minimum signless Laplacian coefficients in the set $\{B_1(n-5, 0, 0, 0, 0), B_2(n-6, 0, 0, 0, 0, 0), B_3(n-4, 0, 0, 0), B_4(n-5, 0, 0, 0, 0)\}$.

Moreover,

$$\begin{aligned}
& Q(B_8(n-7, 0, 0, 0, 0, 0, 0), x) - Q(B_6(n-6, 0, 0, 0, 0, 0), x) \\
&= x(x-1)^{n-8}[x^5 - (n+4)x^4 + (6n+1)x^3 - (11n-6)x^2 + 3(2n+1)x - n]. \quad (5)
\end{aligned}$$

$$\begin{aligned}
& Q(B_6(n-6, 0, 0, 0, 0, 0), x) - Q(B_7(n-5, 0, 0, 0, 0), x) \\
&= x(x-1)^{n-6}[(n-4)x^3 - (7n-28)x^2 + (12n-43)x - 3n]. \quad (6)
\end{aligned}$$

Thus $B_7(n-5, 0, 0, 0, 0)$ has minimum signless Laplacian coefficients in the set $\{B_6(n-6, 0, 0, 0, 0, 0), B_7(n-5, 0, 0, 0, 0), B_8(n-7, 0, 0, 0, 0, 0, 0)\}$.

For convenience, write $B_3(n-4, 0, 0, 0)$ as B_n^1 , $B_7(n-5, 0, 0, 0, 0)$ as B_n^2 .

5 The graphs which have minimum signless Laplacian coefficients in $\mathcal{B}^1(n)$ and $\mathcal{B}^2(n)$

Theorem 5.1 *In the set $\mathcal{B}^1(n)$, for $G \in \mathcal{B}^1(n)$, $G \not\cong B_n^1$, $\varphi_i(G) \geq \varphi_i(B_n^1)$, $i = 0, 1, \dots, n$. With equality if and only if either $i \in \{0, 1, n-1, n\}$ when $\overline{G} \cong B(P_2, P_2, P_1)$ or $i \in \{0, 1\}$ otherwise.*

Proof. Let G be an arbitrary graph in $\mathcal{B}^1(n)$. We need to prove after series of transformations, G will become to B_n^1 and B_n^1 has minimum signless Laplacian coefficients in $\mathcal{B}^1(n)$.

Step 1: When there is a non-pendent edge uv which is not on the cycle. By performing the transformation of Definition 3.1 to uv , we have $G_{uv} \in \mathcal{B}^1(n)$, and $\varphi_i(G) > \varphi_i(G_{uv})$, $i = 2, 3, \dots, n-1$ by Theorem 3.2.

After performing Step 1 consecutively, it is easy to see that all cut edges are pendent edges in the resulting graph.

Step 2: Since G is a bicyclic graph, there are two minimal cycles C_1, C_2 in G . For $u, v, w \in V(C_1)$ and satisfy the claim of Definition 3.3, if $|V(C_1)| \geq 5$, and $|V(C_2) \cap V(C_1)| \geq 3$, by the assumption $|V(C_2) \cap V(C_1)| = \min \{k, l, m\}$, we have $|V(C_2)| \geq 5$. Then we perform the transformation of Definition 3.3 to $u, v, w \in V(C_1) \cap V(C_2)$, and obtain G' , by Theorem 3.4, $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n$, and the lengths of the cycles of G are decreased by 2.

If $|V(C_1)| \geq 5$, and $|V(C_2) \cap V(C_1)| \leq 3$. Assume u, v, w satisfy the claim of Definition 3.3 and at most one of $\{u, v, w\}$ belongs to $V(C_1) \cap V(C_2)$, then we perform the transformation of Definition 3.3 to u, v, w , and obtain G' , by Theorem 3.4, $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n$, and the length of C_1 is decreased by 2, while the length of C_2 keeps unchanged.

Therefore, after taking Step 2 consecutively, we obtain five types of graphs which has been discussed from Lemma 4.1 and Lemma 4.2, then by comparing the signless Laplacian polynomials of the resulting five extremal graphs, it is easy to see that B_n^1 has minimum signless Laplacian coefficients in $\mathcal{B}^1(n)$ by equations (1)-(4). ■

By Theorem 1.3 and Theorem 5.1, we obtained the following corollary.

Corollary 5.2 *In the set of all n -vertex bicyclic graphs in $\mathcal{B}^1(n)$, B_n^1 is the unique graph with the minimal IE .*

Theorem 5.3 *In the set $\mathcal{B}^2(n)$, for $G \in \mathcal{B}^2(n)$, $G \not\cong B_n^2$, $\varphi_i(G) \geq \varphi_i(B_n^2), i = 0, 1, \dots, n$. With equality if and only if either $i \in \{0, 1, n-1, n\}$ when $\overline{G} \cong B(P_2, P_2, P_2)$ or $i \in \{0, 1\}$ otherwise.*

The proof is similar to the proof of Theorem 5.1.

By Theorem 1.3 and Theorem 5.3, we obtained the following corollary.

Corollary 5.4 *In the set of all n -vertex bicyclic graphs in $\mathcal{B}^2(n)$, B_n^2 is the unique graph with the minimal IE .*

From Corollary 5.2 and Corollary 5.4 we immediately get the following result.

Corollary 5.5 *If $n \leq 30$, then for $G \in \mathcal{B}(n)$ we have $IE(G) \geq IE(B_n^2)$, with equality if and only if $G \cong B_n^2$. If $n \geq 31$, then for $G \in \mathcal{B}(n)$ we have $IE(G) \geq IE(B_n^1)$, with equality if and only if $G \cong B_n^1$.*

Proof. From Corollary 5.2 and Corollary 5.4, for $G \in \mathcal{B}(n)$ we have $IE(G) \geq \min \{IE(B_n^1), IE(B_n^2)\}$, with equality if and only if $G \cong B_n^1$ or $G \cong B_n^2$.

From $Q(B_n^1, x)$ and $Q(B_n^2, x)$ in Section 4, we have

$$IE(B_n^1) = (n - 4) + \sqrt{2} + \sqrt{\alpha_1} + \sqrt{\alpha_2} + \sqrt{\alpha_3},$$

$$IE(B_n^2) = (n - 6) + 2\sqrt{2} + \sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3},$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3$ are the roots of $x^3 - (n + 4)x^2 + 4nx - 8 = 0$, $\beta_1 \geq \beta_2 \geq \beta_3$ are the roots of $x^3 - (n + 4)x^2 + (5n - 2)x - 3n = 0$.

For $n \leq 30$, by Matlab 7.0 it is easy to see $IE(B_n^1) > IE(B_n^2)$ holds.

For $n \geq 31$, it is easy to see that $n \leq \alpha_1 \leq n + 0.01$, $3.93 \leq \alpha_2 \leq 4$, $0 \leq \alpha_3 \leq 0.066$, and $n - 1 \leq \beta_1 \leq n - 0.995$, $4.27 \leq \beta_2 \leq 4.31$, $0.697 \leq \beta_3 \leq 0.726$, and $0.5899 \leq \sum_{i=1}^3 (\sqrt{\beta_i} - \sqrt{\alpha_i}) \leq 1$.

Then we have $IE(B_n^2) - IE(B_n^1) = \sqrt{2} - 2 + \sum_{i=1}^3 (\sqrt{\beta_i} - \sqrt{\alpha_i}) \geq 0$. ■

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